PHYS 232 – Assignment #1

Due Wednesday, Jan. 24 @ 11:00

1. Evaluate the following integral:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx$$

Both m and n are integers. Consider the cases $m \neq n$, $m = n \neq 0$, and $m = n = 0$. *Hint*: Rewrite the trigonometric functions in terms of complex exponentials $e^{\pm jx}$ before evaluating the integral.

2. The Fourier series of a periodic function $f(x)$ that has period 2π such that $f(x) = f(x + 2\pi)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

In class we showed that $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$. Show that the coefficients of the sine terms are given by $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$.

3. Below you are given a function on the interval $-\pi < x < \pi$. Sketch several periods of the corresponding periodic function of period 2π. Find the Fourier series of the function.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Use **software** to plot your Fourier expansion of the function. Keep terms up to $\cos 7x$ and $\sin 7x$. I recommend using a Jupyter notebook and Python to make the required plots, however, you can use whatever works best be for you.

This problem is for you own practice... **It won't be graded**.

Below you are given a function on the interval $-\pi < x < \pi$. Sketch several periods of the corresponding periodic function of period 2π. Find the Fourier series of the function.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$$

This next part is just for fun...

In PHYS 231 we briefly looked at low- and high-pass filters made from a single resistor R and a single capacitor C. If a signal of the form $v_{\rm in} = V_0 \sin \omega t$ is passed through the filter, the resulting output is:

Low-pass

$$v_{\rm out} = \frac{V_0}{\sqrt{1 + (\omega/\omega_{\rm c})^2}} \sin(\omega t + \phi_{\rm lp}) \tag{1}$$

High-pass

$$v_{\rm out} = \frac{V_0 (\omega/\omega_{\rm c})}{\sqrt{1 + (\omega/\omega_{\rm c})^2}} \sin(\omega t + \phi_{\rm hp}) \tag{2}$$

where $\omega_{\rm c} = 1/(RC)$, $\tan \phi_{\rm lp} = -\omega/\omega_{\rm c}$, and $\tan \phi_{\rm hp} = +\omega_{\rm c}/\omega$. The variable $\omega_{\rm c}$ is called the corner frequency. For example, the low-pass filter will pass frequencies below $\omega_{\rm c}$ and attenuate signals above $\omega_{\rm c}$. The high-pass filter does the opposite. An equivalent set of filter equations for a periodic function of position is:

$$f_{\rm in}(x) = f_0 \sin(kx)$$

Low-pass

$$f_{\rm out}(x) = \frac{f_0}{\sqrt{1 + (k/k_{\rm c})^2}} \sin(kx + \phi_{\rm lp}) \tag{3}$$

High-pass

$$f_{\rm out}(x) = \frac{f_0 (k/k_{\rm c})}{\sqrt{1 + (k/k_{\rm c})^2}} \sin(kx + \phi_{\rm hp}) \tag{4}$$

where $k = 2\pi/\lambda$ is referred to as a wavenumber and $k_{\rm c}$ is a critical wavenumber set by the filter. The phase shifts due to the filter are given by $\tan \phi_{\rm lp} = -k/k_{\rm c}$, and $\tan \phi_{\rm hp} = +k_{\rm c}/k$.

Let's take, for example, the Fourier series of the square wave discussed in class:

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(nx) \tag{5}$$

In this series, the amplitude of each sinusoidal term in the sum is $1/n$ and the wavenumber is $k = n$. If this function was passed through the low-/high-pass filter, the resulting output would be:

Low-pass

$$f_{\text{out}}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \frac{1}{\sqrt{1 + (n/k_{\text{c}})^2}} \sin(nx + \phi_{\text{lp}}) \tag{6}$$

High-pass

$$f_{\text{out}}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1/k_{\text{c}}}{\sqrt{1 + (n/k_{\text{c}})^2}} \sin(nx + \phi_{\text{hp}}) \tag{7}$$

where $\tan\phi_{\text{lp}} = -n/k_{\text{c}}$ and $\tan\phi_{\text{hp}} = +k_{\text{c}}/n$. Notice that the Fourier series representation of $f(x)$ has allowed us to calculate the response of function of position to a filter whose affect is dependent on wavenumber (or, if you prefer, wavelength). The figures below show plots of Eqs. (5), (6), and (7) keeping terms up to $n = 1001$ and using $k_{\text{c}} = 2$.

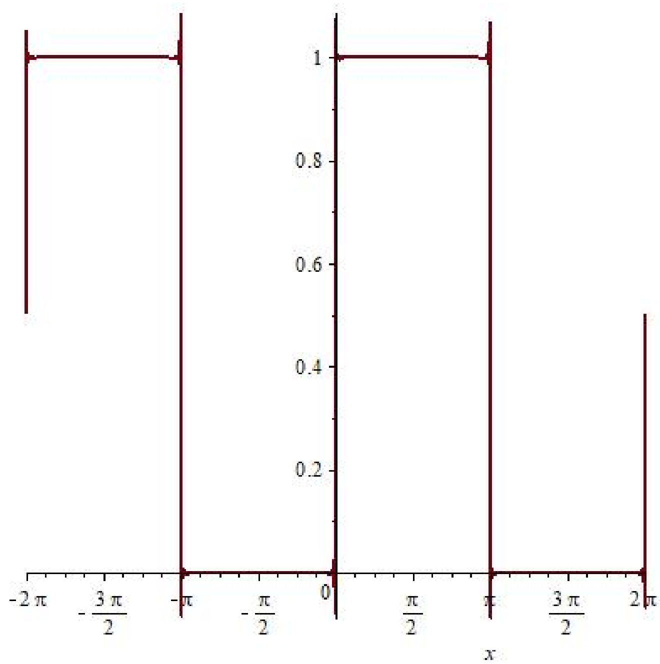

Figure 1: Fourier series of a square wave keeping terms up to $n = 1001$.

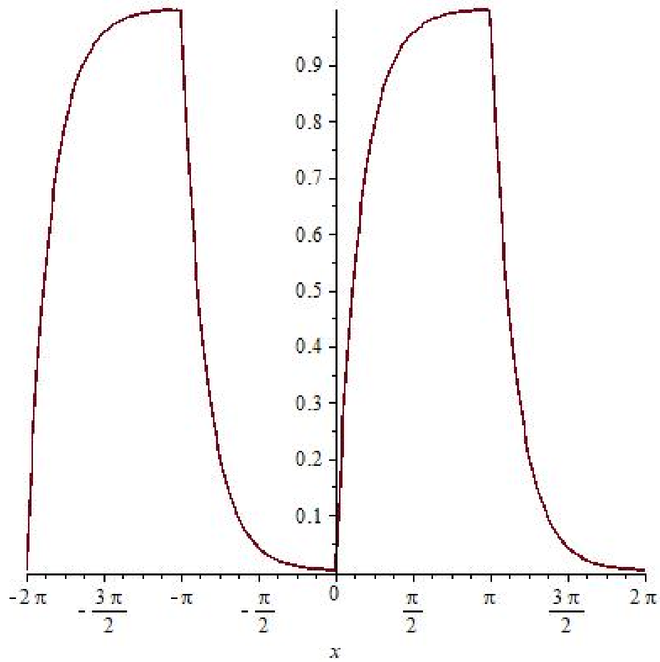

Figure 2: The result of passing the square wave through a low-pass filter with $k_c = 2$.

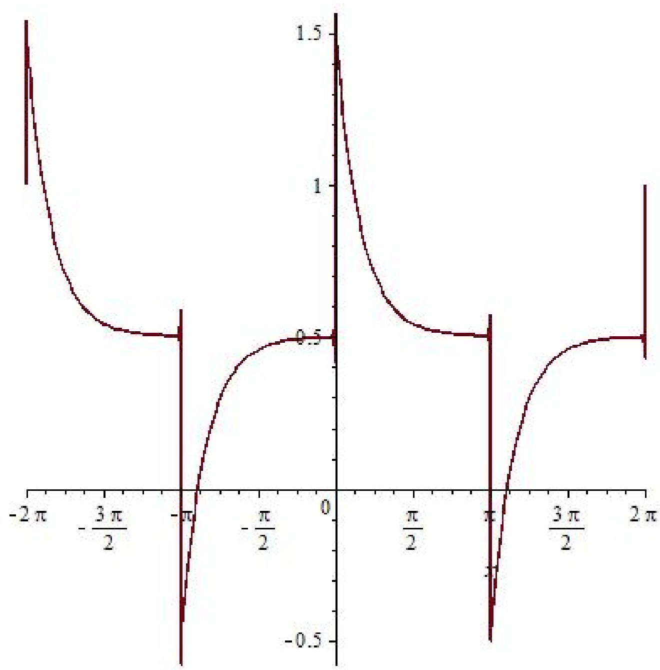

Figure 3: The result of passing the square wave through a high-pass filter with $k_c = 2$.

For the case of the square wave, you may be able to, relatively easily, calculate the response of the function of position to a simple filter without having to use a Fourier series. However, imagine a more complicated function like a triangle wave or the absolute value of a sine wave. It would be much easier the first construct the Fourier series and then apply the result of the filter to each term of the series to determine the shape of the signal at the filter output.

If you're looking for something fun to do, see how the Fourier series that you calculated in problem 3 responds to the low- and high-pass filter. What if the signal was first passed through the low-pass and then the high-pass filter? The fun never ends...